Permanents of Incidence Matrices

By Paul J. Nikolai

1. Introduction. This paper describes the evaluation of the permanents of certain incidence matrices using the UNIVAC Scientific 1103A Computer.

The *permanent* of an n by n matrix $A = [a_{ij}]$ with elements in a commutative ring is defined by the relation

$$\text{per}\,(A) = \sum a_{1i_1}\, a_{2i_2} \cdots a_{ni_n}$$

where the sum extends over all $n!$ permutations of $1, 2, \cdots, n$. The permanent is thus similar in definition to the determinant and suggests a theory for permanents analogous to that for determinants. No such theory is available, however, since permanents do not obey the analogue of the basic multiplicative law of determinants

$$\det\,(AB) = \det\,(A)\,\det\,(B).$$

The analogue of the Laplace expansion is easily shown to remain valid for the permanent, but for computational purposes it is of little help. The calculations described in this paper were made practicable by a computational device given by H. J. Ryser. I shall state Ryser's result, which has heretofore not been published, as a theorem.

THEOREM 1. Denote by A_r a matrix obtained from A by replacing r of the column vectors of A by zero vectors. Denote by $S(A_r)$ the product of the row sums of A_r. Then

$$\text{(1.1)} \quad \text{per}\,(A) = S(A) - \sum_{C_1^n} S(A_1) + \sum_{C_2^n} S(A_2) - \cdots + (-1)^{n-1} \sum_{C_{n-1}^n} S(A_{n-1})$$

where each sum extends over all the C_r^n ways of forming A_r.

Proof. The proof is based on the Principle of Inclusion and Exclusion or Sieve of Sylvester [2].

Consider an element of the form

$$a = a_{1j_1}\, a_{2j_2} \cdots a_{nj_n}$$

where the j_i may assume any of the values $1, 2, \cdots, n$. Let r denote the number of these integers not among $j_1, j_2, \cdots, j_n$. Suppose $r > 0$. Then a appears in (1.1)

$$1 \text{ time in } S(A),$$

$$C_1^r \text{ times in } \sum S(A_1),$$

$$C_2^r \text{ times in } \sum S(A_2),$$

$$\cdots$$

$$C_r^r \text{ times in } \sum S(A_r)$$

Received November 6, 1959.

or

$$1 - C_1^r + C_2^r - \cdots + (-1)^r C_r^r = 0$$

times in all.

On the other hand, if $j_1, j_2, \cdots, j_n$ is a permutation of $1, 2, \cdots, n$, $r = 0$ and a appears in (1.1) exactly once.

One obvious advantage in using Theorem 1 is that the number of summands required to calculate per (A) is reduced from $n!$ to 2^n. Other features of this device are noted in Section 5.

2. The v, k, λ Problem. Let it be required to arrange v elements into v sets such that every set contains exactly k distinct elements and such that every pair of sets has exactly λ elements in common, $0 < \lambda < k < v$. This problem is referred to as the v, k, λ *problem* and the resulting arrangement is called a v, k, λ *configuration* or *symmetric balanced incomplete block design*. For a v, k, λ configuration list the elements $X_1, X_2, \cdots, X_v$ in a row and the sets $T_1, T_2, \cdots, T_v$ in a column. Insert 1 in row i and column j if X_j belongs to set T_i and 0 otherwise. In this way is obtained a v by v matrix A of 0's and 1's called the *incidence matrix* of the v, k, λ configuration. It is not difficult to show that

$$\lambda = k(k - 1)/(v - 1)$$

and that

$$|\det (A)| = k(k - \lambda)^{\frac{v-1}{2}}.$$

Two v, k, λ configurations D_1 and D_2 will be termed *isomorphic* if there is a one-to-one correspondence $X_1 \leftrightarrow X_2 = (X_1)\alpha$ between the elements $\{X_1\}$ of D_1 and the elements $\{X_2\}$ of D_2 and a one-to-one correspondence $T_1 \leftrightarrow T_2 = (T_1)\beta$ between the sets $\{T_1\}$ of D_1 and the sets $\{T_2\}$ of D_2, such that if $X_1 \in T_1$, then $(X_1)\alpha \in (T_1)\beta$. An isomorphism of a design with itself is called a *collineation*.

These combinatorial designs and their associated incidence matrices have been extensively studied. An excellent summary of the problem together with an extensive bibliography can be found in [3].

3. Permanents of Incidence Matrices. A set $R = \{X_1, X_2, \cdots, X_n\}$ will be called a *system of distinct representatives* for the subsets $T_1, T_2, \cdots, T_n$ of the set D in case $X_i \in T_i$ and $X_i \neq X_j$ for $i \neq j$, $i = 1, 2, \cdots, n$. The permanent of an incidence matrix possesses combinatorial significance in that it equals the number of systems of distinct representatives of the class of sets $T_1, T_2, \cdots, T_v$. As pointed out in Section 2, the determinant of the incidence matrix of a v, k, λ configuration is an elementary function of v, k, and λ alone. It seemed of interest to know whether or not the permanent possesses a similar property. Before trying calculations on UNIVAC Scientific, it seemed clear that per (A) could not be a simple function of v, k, and λ but it remained an open question whether or not non-isomorphic designs having the same parameters v, k, and λ could possess unequal permanents. The answers to these questions would be of greater interest in the case of finite projective planes with $n + 1$ points per line which are v, k, λ designs with

$v = n^2 + n + 1$, $k = n + 1$, and $\lambda = 1$. Unfortunately, the first case of non-isomorphic planes arises for $n = 9$ with incidence matrices of order 91. Present theory does not permit calculation of the permanent of an incidence matrix of this size. There are known to be no instances of non-isomorphic v, k, λ configurations for $v < 15$. Different 15, 7, 3 designs, however, do exist, and the permanents of their incidence matrices were easily calculated on UNIVAC Scientific.

Nandi [1] has constructed all 15, 7, 3 designs. There are five non-isomorphic examples. Study of these five designs revealed that exactly two which Nandi denotes by $(\gamma\gamma')$ and $(\alpha_1\alpha_1')_1$ possess a collineation of order 7 fixing one element (and one set) and permuting the remaining 14 elements (sets) in two cycles of length 7. For both designs let $X; A_1, A_2, \cdots, A_7; B_1, B_2, \cdots, B_7$ denote the 15 elements. Each design has a set $\{A_1, A_2, \cdots, A_7\}$ fixed by the collineation $(X)\ (A_1A_2 \cdots A_7)$ $(B_1B_2 \cdots B_7)$. $(\gamma\gamma')$ and $(\alpha_1\alpha_1')_1$ can be displayed in the form

$$\begin{array}{lcl}
\{A_1, A_2, A_3, A_4, A_5, A_6, A_7\} & & \{A_1, A_2, A_3, A_4, A_5, A_6, A_7\} \\
\{X, A_1, A_2, A_4, B_1, B_2, B_4\} & & \{X, A_1, A_2, A_4, B_1, B_3, B_4\} \\
\{X, A_2, A_3, A_5, B_2, B_3, B_5\} & & \{X, A_2, A_3, A_5, B_2, B_4, B_5\} \\
\cdots & & \cdots \\
\{X, A_7, A_1, A_3, B_7, B_1, B_3\} & \text{and} & \{X, A_7, A_1, A_3, B_7, B_2, B_3\} \\
\{A_1, A_2, A_4, B_3, B_5, B_6, B_7\} & & \{A_1, A_2, A_4, B_2, B_5, B_6, B_7\} \\
\{A_2, A_3, A_5, B_4, B_6, B_7, B_1\} & & \{A_2, A_3, A_5, B_3, B_6, B_7, B_1\} \\
\cdots & & \cdots \\
\{A_7, A_1, A_3, B_2, B_4, B_5, B_6\} & & \{A_7, A_1, A_3, B_1, B_4, B_5, B_6\}
\end{array}$$

respectively. Thus their corresponding incidence matrices appear as follows reflecting the collineation:

$$\begin{array}{ccccccccccccccc}
0&1&1&1&1&1&1&1&0&0&0&0&0&0&0\\
1&1&1&0&1&0&0&0&1&1&0&1&0&0&0\\
1&0&1&1&0&1&0&0&0&1&1&0&1&0&0\\
1&0&0&1&1&0&1&0&0&0&1&1&0&1&0\\
1&0&0&0&1&1&0&1&0&0&0&1&1&0&1\\
1&1&0&0&0&1&1&0&1&0&0&0&1&1&0\\
1&0&1&0&0&0&1&1&0&1&0&0&0&1&1\\
1&1&0&1&0&0&0&1&1&0&1&0&0&0&1\\
0&1&1&0&1&0&0&0&0&0&1&0&1&1&1\\
0&0&1&1&0&1&0&0&1&0&0&1&0&1&1\\
0&0&0&1&1&0&1&0&1&1&0&0&1&0&1\\
0&0&0&0&1&1&0&1&1&1&1&0&0&1&0\\
0&1&0&0&0&1&1&0&0&1&1&1&0&0&1\\
0&0&1&0&0&0&1&1&1&0&1&1&1&0&0\\
0&1&0&1&0&0&0&1&0&1&0&1&1&1&0
\end{array}
\qquad
\begin{array}{ccccccccccccccc}
0&1&1&1&1&1&1&1&0&0&0&0&0&0&0\\
1&1&1&0&1&0&0&0&1&0&1&1&0&0&0\\
1&0&1&1&0&1&0&0&0&1&0&1&1&0&0\\
1&0&0&1&1&0&1&0&0&0&1&0&1&1&0\\
1&0&0&0&1&1&0&1&0&0&0&1&0&1&1\\
1&1&0&0&0&1&1&0&1&0&0&0&1&0&1\\
1&0&1&0&0&0&1&1&1&1&0&0&0&1&0\\
1&1&0&1&0&0&0&1&0&1&1&0&0&0&1\\
0&1&1&0&1&0&0&0&0&1&0&0&1&1&1\\
0&0&1&1&0&1&0&0&1&0&1&0&0&1&1\\
0&0&0&1&1&0&1&0&1&1&0&1&0&0&1\\
0&0&0&0&1&1&0&1&1&1&1&0&1&0&0\\
0&1&0&0&0&1&1&0&0&1&1&1&0&1&0\\
0&0&1&0&0&0&1&1&0&0&1&1&1&0&1\\
0&1&0&1&0&0&0&1&1&0&0&1&1&1&0
\end{array}$$

The permanent of each matrix is the sum of the equal minors belonging to the fixed row which represents the fixed set. This row and one column containing a 1 in this row can be deleted reducing the order of the matrix by one. The permanent of the reduced matrix is computed and multiplied by the number of 1's per row to yield the permanent of the incidence matrix.

4. Results. The computational advantage offered by the collineations made the choice of $(\gamma\gamma')$ and $(\alpha_1\alpha_1')_1$ for a first trial a natural one. The permanents of the incidence matrices of these designs were found to be 24, 601, 472 and 24, 567, 424 respectively. Thus nonisomorphic designs having the same parameters v, k, and λ may have unequal permanents, so that the permanent is not a function of v, k, and λ alone.

The program for evaluation of these permanents turned out to be an efficient one requiring about four minutes of computer time for each matrix. The remaining 15, 7, 3 designs were also run with the following results:

Design	Permanent
$(\alpha_1\alpha_1')_2$	24,572,288
$(\alpha_2\alpha_2')$	24,567,424
$(\beta_1\beta_1')$	24,582,016

As computer time became available it was decided to calculate the permanents of the incidence matrices of all cyclic designs with prime v and $v < 23$. Cyclic designs with their simple structure and with v a prime might have yielded possible clues to a formula for the permanent in the cyclic case or to possible divisibility properties. Unfortunately no distinct cyclic designs with the same parameters arise for $v < 31$. Results of this survey are given as follows:

v	k	λ	per (A)
7	3	1	24
7	4	2	144
11	5	2	12,105
11	6	3	75,510
13	4	1	3,852
13	9	6	64,803,969
19	9	4	142,408,674,153
19	10	5	952,709,388,762

In addition, the permanent of the incidence matrix of the projective plane of order 4, a cyclic 21, 5, 1 design, was computed and found to be 18, 534, 400. Here, as with two of the 15, 7, 3 designs, the use of a collineation reduced computation time by more than one half. This was a significant saving, better than four hours of computer time.

5. Description of the Code for the UNIVAC Scientific Computer. Ones complement binary arithmetic, two address logic, and an extensive array of logical instructions together with equation (1.1) applied to 0,1 matrices contributed to a short, fast computer code for UNIVAC Scientific. The v rows of the square matrix A of 0's and 1's were stored in the higher order v stages of v consecutive storage cells. An index r, $0 \leqq r \leqq 2^v - 1$, counted the number of A_r's formed, and served as a logical multiplier in forming A_r. If no row of A_r were zero, $S(A_r)$ was calculated and added or subtracted from the accumulated sum according as the number of 1's in the binary representation of r was even or odd. A_{r-1} was formed next and the calculation continued. The magnitude of the final sum was then per (A).

All loops of the code were checked using the v by v matrix S of all 1's. per $(S) = v!$. Machine accuracy was checked by running each calculation twice in every case.

6. Acknowledgments. This work was begun while the author was employed by Remington Rand UNIVAC, Division of Sperry Rand Corporation, St. Paul, Minnesota. Calculations were completed at Wright Air Development Center, Wright-Patterson Air Force Base, Ohio. The author expresses his thanks to Dr. Ernest T. Parker for suggesting the problem, to Remington Rand UNIVAC and to the Digital Computation Branch at Wright Air Development Center for providing time on their UNIVAC Scientific computers.

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On the Numerical Treatment of Heat Conduction Problems with Mixed Boundary Conditions

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Abstract. The two-dimensional problem of heat conduction in a rectangle where the temperature is prescribed over a portion of the boundary while the temperature gradient is prescribed over the remainder of the boundary, may be treated numerically by replacing the differential equation of heat conduction and the equations expressing the given initial and boundary conditions by their difference analogs

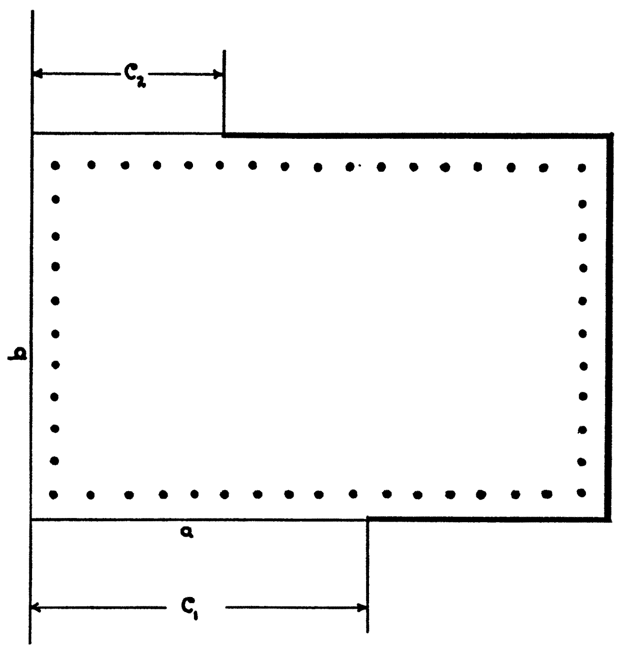

Fig. 1.—Rectangular domain with "mixed" boundary conditions.

Received September 15, 1959.